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Cut-free formulations for a quantified logic of here and there

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This paper is dedicated to N.A. Shanin who have taught me and his other students not only logic but also the remarkable lesson of love of truth, of science and of clear thinking.

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ABSTRACT

A predicate extension $SQHT^=$ of the logic of here-and-there was introduced by V. Lifschitz, D. Pearce, and A. Valverde to characterize strong equivalence of logic programs with variables and equality with respect to stable models. The semantics for this logic is determined by intuitionistic Kripke models with two worlds (*here* and *there*) with constant individual domain and decidable equality. Our sequent formulation has special rules for implication and for pushing negation inside formulas. The soundness proof allows us to establish that $SQHT^=$ is a conservative extension of the logic of weak excluded middle with respect to sequents without positive occurrences of implication. The completeness proof uses a non-closed branch of a proof search tree. The interplay between rules for pushing negation inside and truth in the "there" (non-root) world of the resulting Kripke model can be of independent interest. We prove that existence is definable in terms of remaining connectives.

1. Introduction

A predicate extension $SQHT^=$ of the logic of here-and-there was introduced in [2] to characterize the notion of strong equivalence of logic programs with variables and equality with respect to stable models. The semantics for this logic is determined by intuitionistic Kripke models $M = \langle W, R, D\rangle$ where $W = \{h, t\}$ consists of two worlds (h for *here* and t for *there*) with R reflexive and hRt, and D a non-empty constant individual domain. Equality is interpreted in a decidable way:

$$t \models a = b \text{ iff } h \models a = b.$$

The logic can be axiomatized [2] by adding to the intuitionistic predicate logic three axiom schemata:

$$HOS: \quad F \vee F \to G \vee \neg G,$$
$$SQHT: \quad \exists x(F(x) \to \forall x F(x)),$$
$$DE: \quad \forall x \forall y(x = y \vee x \neq y).$$

This makes derivable the weak law of excluded middle

$$WEM: \quad \neg F \vee \neg\neg F$$

and the standard classical rules of moving quantifiers inside $\vee$, $\neg$, $\to$, for example

$$(\forall x F(x) \to p) \leftrightarrow \exists x(F(x) \to p); \ (p \to \exists x F(x)) \leftrightarrow \exists x(p \to F(x)).$$

Since the world t is the "endworld" in the order R, the semantics of $t \models F$ is classical.

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Two cut-free formulations of the logic $SQHT^{=}$ are presented below. A sequent formulation derives sequents $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ and postulates special rules for implication and for pushing negation inside formulas. It is presented in Section 2. The soundness proof allows us to establish in Section 4 that $SQHT^{=}$ is a conservative extension of the logic of weak excluded middle with respect to sequents without positive occurrences of implication. This extends the results of [3] by adding quantifiers and equality. The completeness proof is a modification of a familiar non-effective cut-elimination proof for first-order logic using a non-closed branch of a proof search tree. The interplay between rules for pushing negation inside and truth in the "there" (non-root) world of the resulting Kripke model can be of independent interest. For the propositional fragment the proof search procedure terminates and allows us to construct a conjunctive normal form in way very close to [1].

Another formulation presented in Section 5 is much closer to familiar Kripke-style semantic tableaux [4,5]. This version has been considerably simplified according to a suggestion by van Benthem.

In Section 3 we prove the definability of existence in terms of the remaining connectives. van Benthem [6] investigated this definition and its consequences.

Most of the results presented here have been reported at the Stanford logic seminar. I thank Vladimir Lifschitz for stating the questions answered here and the participants of the logic seminar for useful discussion.

2. A sequent formulation $HTG^{=}$

Derivable objects are ordinary sequents $\Gamma \Rightarrow \Delta$ where Γ, Δ are finite sets of formulas. Negation $\neg$ is a separate connective: although it is definable from $\to, \bot$, it plays a prominent part below.

2.1. The propositional fragment

Axioms: $A, \Gamma \Rightarrow \Delta, A$; $A, \neg A, \Gamma \Rightarrow \Delta$.
Inference Rules HTp
Familiar classical rules for $\&, \vee$ and new rules for $\to$:

$$\frac{\neg A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A, \neg B \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \to\Rightarrow$$

$$\frac{A, \Gamma \Rightarrow \Delta, B \quad \neg B, \Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, A \to B} \Rightarrow\to .$$

The rules for moving negation inside based on the equivalences

$$\neg(A\&B) \leftrightarrow (\neg A \vee \neg B);\ \neg(A \vee B) \leftrightarrow (\neg A\&\neg B);\ \neg(A \to B) \leftrightarrow (\neg\neg A\&\neg B) \tag{1}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A\&B)} \qquad \frac{\neg A, \Gamma \Rightarrow \Delta \quad \neg B, \Gamma \Rightarrow \Delta}{\neg(A\&B), \Gamma \Rightarrow \Delta}$$

$$\frac{\neg B, \Gamma \Rightarrow \Delta, \neg A}{\neg(A \to B), \Gamma \Rightarrow \Delta} \qquad \frac{\neg A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \to B)}$$

$$\frac{\neg A, \neg B, \Gamma \Rightarrow \Delta}{\neg(A \vee B), \Gamma \Rightarrow \Delta} \qquad \frac{\neg A, \Gamma \Rightarrow \Delta \quad \neg B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}$$

$\neg\neg$ rules:

$$\frac{\Gamma \Rightarrow \Delta, \neg A}{\neg\neg A, \Gamma \Rightarrow \Delta} \qquad \frac{\neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

Lemma 1. *The rules HTp are sound and complete for the propositional fragment of SQHT.*

Proof. For soundness note that all equivalences (1) and $\neg\neg$ rules are provable in the logic of weak excluded middle. The rule $\Rightarrow\to$ is derived using HOS; *int* indicates intuitionistic derivability:

$$\frac{\dfrac{A, \Gamma \Rightarrow \Delta, B}{A, \Gamma \Rightarrow \Delta, A \to B}\,int \quad A \to B, \Gamma \Rightarrow \Delta, A \to B \quad \dfrac{\neg B, \Gamma \Rightarrow \Delta, \neg A}{\neg B, \Gamma \Rightarrow \Delta, A \to B}\,int}{\Gamma \Rightarrow \Delta, A \to B}\,\text{HOS}$$

The remaining rules are intuitionistically valid. In particular the implication

$$((\neg A \to C)\&(C \vee A \vee \neg B)\&(B \to C) \to ((A \to B) \to C) \tag{2}$$

justifying the rule $\to\Rightarrow$ is intuitionistic.

To establish completeness, note that each of the rules applied bottom-up reduces either the number of binary connectives or the total number of connectives. Hence the process of such bottom-up applications beginning with some formula F terminates after a finite number of steps either in a closed tree (all branches terminate in axioms) or in a tree with at least one non-closed leaf (terminal sequent to which no rule can be applied) consisting of atomic formulas and their negations.

Take any such non-axiom sequent $\Gamma \Rightarrow \Delta$. Define a Kripke model with two worlds $\{h, t\}$. Define all atomic formulas $A \in \Gamma$ to be true in the world h (and hence in t): $h \models A$. Define $h \not\models A$ for the remaining atomic formulas A. For atomic A with $A \notin \Gamma$ define $t \models A$, if $(\neg A) \in \Delta$ and $t \not\models A$ otherwise.

In this way all antecedent formulas evaluate to $\top$ in h, and all succedent formulas evaluate to $\bot$, so $\Gamma \Rightarrow \Delta$ is refuted. Detailed verification is given below after quantifier and equality rules are introduced. Since all our inference rules are invertible (the implication inverse to (2) requires HOS) the same model refutes the original formula F. $\square$

Note. Since the inference rules are sound and invertible, the original formula F is equivalent to the conjuction of translations $\&\Gamma \to \vee\Delta$ of all non-closed leaves of the proof search tree. This conjunctive normal form is similar to [1].

2.2. Quantifier and equality rules

In addition to ordinary classical rules for both quantifiers, for example

$$\frac{\Gamma \Rightarrow \Delta, A(b)}{\Gamma \Rightarrow \Delta, \forall x A(x)} \Rightarrow\forall \qquad \frac{A(t), \forall x A(x), \Gamma \Rightarrow \Delta}{\forall x A(x), \Gamma \Rightarrow \Delta} \forall\to$$

we postulate the following rules for moving quantifiers $Q \in \{\forall, \exists\}$ inside negation based on the equivalences:

$$\neg QxA \leftrightarrow \bar{Q}x\neg A. \tag{3}$$

$$\frac{\neg A(b), \Gamma \Rightarrow \Delta}{\neg\forall x A(x), \Gamma \Rightarrow \Delta} \qquad \frac{\neg A(t), \neg\exists x A(x), \Gamma \Rightarrow \Delta}{\neg\exists x A(x), \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg A(b)}{\Gamma \Rightarrow \Delta, \neg\exists x A(x)} \qquad \frac{\Gamma \Rightarrow \Delta, \neg A(t), \neg\forall x A(x)}{\Gamma \Rightarrow \Delta, \neg\forall x A(x)}$$

where b is a fresh variable, t is an arbitrary term.

Equality rules

Axioms: $\Gamma \Rightarrow \Delta, t = t$

$$\frac{t = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(t = s)} \neq \frac{\Gamma \Rightarrow \Delta, t = s}{\neg(t = s), \Gamma \Rightarrow \Delta}$$

$$\frac{t = s, A(s), \Gamma \Rightarrow \Delta}{t = s, A(t), \Gamma \Rightarrow \Delta} Eq \qquad \frac{t = s, \Gamma \Rightarrow \Delta, A(s)}{t = s, \Gamma \Rightarrow \Delta, A(t)}$$

and similarly with $t = s$ changed to $s = t$.

Theorem 2. *The rules HTG$^=$ are sound and complete for SQHT$^=$.*

Proof. Soundness is easy. As noticed in the Introduction, the rules for moving quantifiers inside negation as well as $\Rightarrow\forall$ are derivable using the axiom SQHT. The rule ($\neq$) is justified by DE. The remaining quantifier and equality rules are intuitionistic.

The completeness proof follows the familiar pattern with some complications. Given a formula F, let us run a fair proof search procedure. More precisely, apply rules bottom-up so that the saturation conditions below are satisfied for every non-closed branch

$$\mathcal{B} := \Gamma \Rightarrow \Delta$$

where Γ, Δ can be infinite sets of formulas. A branch of the tree is closed if it contains an axiom.

If not all branches of the tree are closed, the tree has a non-closed branch $\mathcal{B}$ by Koenig's Lemma. $\mathcal{B}$ defines a countermodel

$$M = (\{h, t\}, R, U, \models)$$

for F in the way same way as in the propositional case. The universe is the set U of all terms constructed from free variables in $\mathcal{B}$ by functions in F.

Consider first the equality-free case. Define for atomic formulas A

$h \models A$ iff $A \in \Gamma$;

$t \models A$ iff $A \in \Gamma$ or $(\neg A) \in \Delta$

Note that $\models$ is monotonic for atomic A: if $h \models A$ then $t \models A$. Indeed $t \not\models A$ implies $A \notin \Gamma$, hence $h \not\models A$.

Fairness of proof search means that the following saturation conditions are satisfied for every $w \in \{h, t\}$:

1. If $A \in \Gamma$ then $A \notin \Delta$.
2. If $A\&B \in \Gamma$ then $A, B \in \Gamma$.
3. If $A\&B \in \Delta$ then $A \in \Delta$ or $B \in \Delta$.
4. If $A \vee B \in \Delta$ then $A, B \in \Delta$.
5. If $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.
6. If $\forall x Ax \in \Gamma$ then $At \in \Gamma$ for all $t \in U$.
7. If $\forall x Ax \in \Delta$ then $At \in \Delta$ for some $t \in U$.

8. If $\exists xAx \in \Delta$ then $At \in \Delta$ for all $t \in U$.
9. If $\exists xAx \in \Gamma$ then $At \in \Gamma$ for some $t \in U$.
10. If $A \to B \in \Gamma$ then either $(\neg A) \in \Gamma$ or $A, \neg B \in \Delta$ or $B \in \Gamma$.
11. If $A \to B \in \Delta$ then either ($A \in \Gamma$ and $B \in \Delta$) or ($\neg B \in \Gamma$ and $\neg A \in \Delta$).
12. If $\neg(A\&B) \in \Delta$ then $\neg A, \neg B \in \Delta$.
13. If $\neg(A\&B) \in \Gamma$ then $\neg A \in \Gamma$ or $\neg B \in \Gamma$.
14. If $\neg(A \vee B) \in \Gamma$ then $\neg A, \neg B \in \Gamma$.
15. If $\neg(A \vee B) \in \Delta$ then $\neg A \in \Delta$ or $\neg B \in \Delta$.
16. If $\neg(A \to B) \in \Gamma$ then $\neg A \in \Delta, \neg B \in \Gamma$.
17. If $\neg(A \to B) \in \Delta$ then $\neg A \in \Gamma$ or $\neg B \in \Delta$.
18. If $\neg\neg A \in \Gamma$ then $\neg A \in \Delta$.
19. If $\neg\neg A \in \Delta$ then $\neg A \in \Gamma$.
20. If $\neg\forall x A(x) \in \Gamma$ then $\neg A(t) \in \Gamma$ for some t.
21. If $\neg\forall x A(x) \in \Delta$ then $\neg A(t) \in \Delta$ for all suitable t.
22. If $\neg\exists x A(x) \in \Delta$ then $\neg A(t) \in \Delta$ for some t.
23. If $\neg\exists x A(x) \in \Gamma$ then $\neg A(t) \in \Gamma$ for all suitable t.
24. If $\neg(t = s) \in \Gamma$ then $(t = s) \in \Delta$.
25. If $\neg(t = s) \in \Delta$ then $(t = s) \in \Gamma$.
26. if $t = s, A(t) \in \Gamma$ then $A(s) \in \Gamma$.
27. if $t = s \in \Gamma, A(t) \in \Delta$ then $A(s) \in \Delta$,
28. and similarly for $s = t \in \Gamma$. □

Lemma 3. 1. *If $A \in \Gamma$ then $h \models A$.*
2. *If $A \in \Gamma$ then $t \models A$.*
3. *If $A \in \Delta$ then $h \not\models A$.*
4. *If $(\neg A) \in \Delta$ then $t \models A$.*

Proof. 1–4 are proved by simultaneous induction on the number of binary connectives in A with subsidiary induction on the number of negations.

Induction base. A is atomic. If $A \in \Gamma$, then $h \models A,\ t \models A$ by definition of the model M. If $A \in \Delta$ then $A \notin \Gamma$ since we took a non-closed branch, hence $h \not\models A$. If $\neg A \in \Delta$ then $t \models A$ by definition of the model M.

Induction step. We consider cases depending of the main connective of the formula A. If the main connective is different from $\to, \neg$, use the induction assumption and saturation conditions.

Let $A \equiv (B \to C)$. If $A \in \Gamma$ then either $\neg B \in \Gamma$ or $C \in \Gamma$ or $B, \neg C \in \Delta$. Then by Induction Hypothesis either $h, t \models \neg B$ or $h, t \models C$ or $h \not\models B, \neg C$ and $t \models C$. In the first two cases $h, t \models (B \to C)$. In the third case, for every $w \in \{h, t\}$ either $w \not\models B$ or $w \models C$, so $w \models (B \to C)$.

If $A \in \Delta$ then either $B \in \Gamma,\ C \in \Delta$ or $\neg C \in \Gamma, \neg B \in \Delta$. In the first case by induction hypothesis $h \models B,\ h \not\models C$, hence $h \not\models (B \to C)$. In the second case by induction hypothesis $h \models \neg C,\ t \models B$, hence $t \models B, \neg C$, so $t \not\models (B \to C)$.

Let now A begin with a negation. Consider subcases depending of the form of the negated formula.

Let $A \equiv \neg B$ for atomic B. If $A \in \Gamma$ then $A \notin \Delta,\ B \notin \Gamma$, since we consider non-closed branch, hence $h, t \not\models B$, so $h \models A$. If $A \in \Delta$, then $t \models B$ by the definition of the model M, hence $h, t \not\models \neg B$.

Let $A \equiv \neg\neg B$. If $A \in \Gamma$ then $\neg B \in \Delta$ and by induction hypothesis $t \models B$, hence $h, t \models A$. If $A \in \Delta$ then $\neg B \in \Gamma$ by induction hypothesis $h, t \models \neg B$ hence $h, t \not\models B$. If $(\neg A) \in \Delta$, then $(\neg B) \in \Delta$ by two applications of the saturation condition for $\neg\neg$, hence by Induction hypothesis $t \models B$, hence $t \models A$.

Let $A \equiv \neg(B\&C)$. If $A \in \Gamma$ then $(\neg B) \in \Gamma$ or $(\neg C) \in \Gamma$. In the first case $h, t \models \neg B$ by inductive hypothesis, hence $h, t \models A$, similarly for the second case. If $A \in \Delta$ then $(\neg B), (\neg C) \in \Delta$, hence $h \not\models \neg B, h \not\models \neg C,\ t \models B, C$ by Induction Hypothesis. So $t \models A$, and $h \not\models A$, the latter because of the first equivalence in (1). If $\neg A \in \Delta$ then $A \in \Gamma$ and $t \models A$ by Induction Hypothesis.

The case $A \equiv \neg(B \vee C)$ is similar.

Let $A \equiv \neg(B \to C)$. If $A \in \Gamma$ then $(\neg B) \in \Delta$ and $(\neg C) \in \Gamma$. Hence $h \not\models \neg B, h, t \models \neg C, t \models B$ by Induction Hypothesis, so $t \not\models (B \to C)$, hence $h, t \models A$.

If $A \in \Delta$ then $(\neg B) \in \Gamma$ or $\neg C \in \Delta$. In the first case $h, t \models \neg B$, hence $h, t \models (B \to C)$. In the second case $t \models C$ by Induction Hypothesis. So $t \models (B \to C)$, and $h \not\models A$. If $\neg A \in \Delta$ then $A \in \Gamma$ and $t \models A$. □

Now completeness for the equality-free case is obvious: either the proof search tree is closed, hence this tree is a derivation by rules *HTG*, or the infinite branch provides a countermodel M for the original formula F.

When equality is present, consider a quotient model. Define the relation $\approx$ on the universe U of the model M by

$$t \approx s \text{ iff } t, s \text{ coincide or } t = s \in \Gamma \text{ or } s = t \in \Gamma.$$

The relation $\approx$ is reflexive and symmetric by definition. It is transitive: if $s = t, t = u \in \Gamma$ then $s = u \in \Gamma$ by saturation for equality rules.

Moreover $\approx$ is a congruence relation with respect to all predicates. Assume first $h \models P\mathbf{t}$ in M where $\mathbf{t} \equiv t_1, \ldots, t_n$, and $\mathbf{t}' \equiv t'_1, \ldots, t'_n$ is given with $\mathbf{t} \approx \mathbf{t}'$, that is $(t'_i \approx t_i) \in \Gamma$ for all $i = 1, \ldots, n$. We have $P\mathbf{t} \in \Gamma$, hence $P\mathbf{t}' \in \Gamma$ by saturation for equality rules.

Assume now $t \models P\mathbf{t}$ because $(\neg P\mathbf{t}) \in \Delta$, and $\mathbf{t} \approx \mathbf{t}'$. Then again $(\mathbf{t} = \mathbf{t}') \in \Gamma$, hence $(\neg P\mathbf{t}') \in \Delta$ by saturation for equality rules, so $t \models P\mathbf{t}'$ as required. Now form a quotient model

$$\tilde{M} :\equiv (\{h, t\}, R, \tilde{U}, \models')$$

in a familiar way by setting

$$\tilde{u} := \{s \in U : s \approx u\}; \qquad \tilde{U} := \{\tilde{u} : u \in U\};$$
$$w \models' P\tilde{\mathbf{t}} :\equiv w \models P\mathbf{t} \text{ in } M.$$

Now the proof of Lemma 3 and hence the completeness proof goes for $\tilde{M}$ without changes. $\square$

3. Definability of existence

$$\exists x Px \leftrightarrow \forall y(\forall x(Px \to Py) \to Py)$$

Proof. The $\Rightarrow$ direction is obvious.

For $\Leftarrow$ argue semantically. Note that the r.h.s. is intuitionistically equivalent to

$$\forall y(\exists x Px \to Py) \to Py).$$

If $\exists x Px$, that is $h \models \exists x Px$, we are done, so assume

$$h \not\models \exists x Px. \tag{4}$$

Assume

$$h \models \forall y((\exists x Px \to Py) \to Py). \tag{5}$$

If $h \models \neg\exists x Px$ then by (5) $h \models \forall y Py$, a contradiction. So assume $t \models \exists x Px$, say

$$t \models Pb.$$

By (4), we have

$$h \models \exists x Px \to Pb. \tag{6}$$

Instantiation of (5) by b gives $h \models Pb$, hence $h \models \exists x Px$. $\square$

For control, here is a derivation. We omit redundant formulas.

Abbreviations:

$$E := \exists x Px; \;\; \alpha := \forall y((E \to Py) \to Py); \;\; \alpha(t) := (E \to Pt) \to Pt.$$

$$\dfrac{\dfrac{\dfrac{d}{\neg(E \to Pa), \alpha \Rightarrow E} \quad \dfrac{E \Rightarrow E \quad \neg Pa \Rightarrow Pa}{\Rightarrow E, E \to Pa, \neg Pa} \quad \dfrac{\text{int}}{Pa \Rightarrow E}}{\alpha(a), \alpha \Rightarrow E}}{\alpha \Rightarrow E}$$

where d is

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg Pb \Rightarrow E, \neg Pb, \neg E}{\neg(E \to Pb) \Rightarrow E, \neg Pb} \quad \dfrac{E \Rightarrow E \quad \neg Pb \Rightarrow \neg Pb}{\Rightarrow E, \neg Pb, E \to Pb} \quad Pb \to E}{\alpha(b) \Rightarrow E, \neg Pb}}{\alpha \Rightarrow E, \neg Pb}}{\alpha \Rightarrow E, \neg E}$$

4. Conservativity over the logic of weak excluded middle

Let $WEM^=$ be an extension of intuitionistic logic with $\neg p \vee \neg\neg p$, and the rules for moving quantifiers inside connectives $\neg, \vee$ and rules for decidable equality.

Theorem 4. *If a sequent $\Gamma \Rightarrow \Delta$ does not contain positive occurrences of $\to$ and is provable in $SQHT^=$ then it is provable in $WEM^=$.*

Proof. If there is no positive $\to$ in the endsequent then its proof by the rules $HTG^=$ does not contain $\Rightarrow\to$.

Hence the translation of this proof back into $SQHT^=$ uses only $WEM^=$. $\square$

5. Hypersequent formulation

5.1. Inference rules

Derivable objects are *hypersequents* $\Gamma \Rightarrow \Delta * \Pi \Rightarrow \Phi$.
Reading: $\Gamma \Rightarrow \Delta$ here or $\Pi \Rightarrow \Phi$ there.
$\Gamma \Rightarrow \Delta$ is called the *h-component*, $\Pi \Rightarrow \Phi$ is the t-component.
Translation into formulas: $\&\Gamma \to \vee\Delta \vee (\&\Pi \to \neg\neg \vee \Phi)$.

Inference rules.
Axioms $A, \Gamma \Rightarrow \Delta, A * \Pi \Rightarrow \Phi, \quad \Gamma \Rightarrow \Delta * A, \Pi \Rightarrow \Phi, A$.
Classical rules for all connectives in the t-component, for example

$$\frac{\Gamma \Rightarrow \Delta * A, \Pi \Rightarrow \Phi, B}{\Gamma \Rightarrow \Delta * \Pi \Rightarrow \Phi, A \to B} \to\Rightarrow_t.$$

Classical rules for all connectives except $\Rightarrow\to$, $\Rightarrow \neg$ in the h-component, for example

$$\frac{\Gamma \Rightarrow \Delta, A(b) * \Pi \Rightarrow \Phi}{\Gamma \Rightarrow \Delta, \forall x A(x) * \Pi \Rightarrow \Phi} \Rightarrow_h \forall, \quad \text{fresh variable } b.$$

Special rule for $\Rightarrow_h\to$:

$$\frac{A, \Gamma \Rightarrow \Delta, B * \Pi \Rightarrow \Phi \quad \Gamma \Rightarrow \Delta * A, \Pi \Rightarrow \Phi, B}{\Gamma \Rightarrow \Delta, A \to B * \Pi \Rightarrow \Phi}.$$

Negation $\neg A$ is defined as $A \to \bot$. If it is treated as a separate connective, the rule $\Rightarrow_h \neg$ can be simplified:

$$\frac{\Gamma \Rightarrow \Delta, *A, \Pi \Rightarrow \Phi}{\Gamma \Rightarrow \Delta, \neg A * \Pi \Rightarrow \Phi}.$$

The remaining rule:

$$\frac{A, \Gamma \Rightarrow \Delta * A, \Pi \Rightarrow \Phi}{A, \Gamma \Rightarrow \Delta * \Pi \Rightarrow \Phi} \text{Transfer}.$$

To add decidable equality, add ordinary equality axioms and rules in both components plus

$$\frac{t = s, \Gamma \Rightarrow \Delta * t = s, \Pi \Rightarrow \Phi}{\Gamma \Rightarrow \Delta * t = s, \Pi \Rightarrow \Phi}.$$

5.2. Soundness and completeness

Theorem 5. *The hypersequent rules are sound.*

Proof. Check that translations of all rules are derivable in $SQHT^{=}$ (with cut). □

Theorem 6. *The rules are complete.*

Proof. Similar to the proof of Theorem 2. □

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